MIDTERM: ALGEBRA II

Date: **24th March 2019**
The Total points is **110** and the maximum you can score is **100** points.

(1) (30 pts) Answer True or False. No justification needed.
   (a) The set $S = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c \in \mathbb{R}, d \text{ is an integer}\}$ is a vector space with usual addition and scalar multiplication.
   (b) The set $T = \{p(x) | \ p(x) \text{ is a polynomial with } \mathbb{R} \text{ coefficients and } p(1) = 0\}$ is a $\mathbb{R}$-vector space with usual addition and scalar multiplication.
   (c) Every homogeneous system of 3 equations and 5 unknowns over $\mathbb{R}$ have infinitely many solutions.
   (d) Every homogeneous system of 5 equations and 3 unknowns over $\mathbb{R}$ have only one solution.
   Let $M(m, n)$ denote the F-vector space of $m \times n$ matrices with entries in F. Let $T : M(3, 2) \oplus M(2, 3) \to M(3, 3)$ be the function $T(A, B) = AB$.
   (e) T is a linear map.
   (f) T is onto.

(2) (10 points) Show that $V = \{(a, b, c, d) : a, b, c, d \in \mathbb{R}, a - b + 2c = 0\}$ is a subspace of $\mathbb{R}^4$. Compute its dimension and find a basis of V?

(3) (15 points) Let $\mathbb{P}_n$ denote the vector space of polynomials with real coefficient of degree at most n. Show that $B = \{x(x-1)(x-2)(x-3), x(x-1)(x-2)(x-4), x(x-1)(x-3)(x-4), x(x-2)(x-3)(x-4), (x-1)(x-2)(x-3)(x-4)\}$ is a basis of $\mathbb{P}_4$. Treating B as an ordered basis of $\mathbb{P}_3$, compute the coordinates of x with respect to B.

(4) (15 points) Find a reduced echelon form of the following matrix using elementary row operations.
$$\begin{bmatrix} 1 & 2 & 1 & 0 & -1 \\ 2 & 2 & 3 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 \end{bmatrix}$$
   Find all solutions to the system of equations:
$$x_1 + 2x_2 + x_3 = -1$$
$$2x_1 + 2x_2 + 3x_3 + x_4 = 0$$
$$x_2 + x_3 + 2x_4 = 1$$

(5) (5+5+10=20 points) Let $\mathbb{P}_n$ denote the vector space of polynomials with real coefficient of degree at most n. Show that $B = \{x^2 - 1, x - 1, x\}$ is a basis of $\mathbb{P}_2$. Show that there exist a linear function $T : \mathbb{P}_2 \to \mathbb{R}^2$ such that
$$T(x^2 - 1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, T(x - 1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, T(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$
   Find the matrix of T with respect to B and the basis $B' = \{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$ of $\mathbb{R}^2$.

(6) (5+15=20 points) Let ϕ be an endomorphism of a vector space V over a field F. Define eigen value and eigen vector of ϕ. Let A and B be nonzero square matrices and D be a diagonal matrix such that $AB = BD$. Show that A has an eigen vector.